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Let this number be g_n.

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The generating function for g_n is

x^a $\quad$ x^{5b}

$$G(x) = (1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)$$

$$= \sum_n g_n x^n \qquad (1 + x^{10} + x^{20} + \cdots)(1 + x^{25} + x^{50} + \cdots)$$

x^{10c} $\quad$ x^{25d}

$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

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Partitions of Integers

$$\begin{aligned} 3 &= 3 \\ &= 2 + 1 \\ &= 1 + 1 + 1 \end{aligned}$$

3 ways to partition 3

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5 ways to partition 4

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Let the number of ways to partition an integer n be $p(n)$.

Then the generating function for $p(n)$ is (assuming $p(0)=1$)

$$P(x) = \sum_{n\geq 0} p(n)\, x^n$$
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A typical term $x^n = x^{k_1} x^{2k_2} x^{3k_3} x^{4k_4} \cdots$

$\Leftrightarrow n = k_1 + 2k_2 + 3k_3 + 4k_4 + \cdots$

Hence $$P(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}$$

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<u>Theorem</u> (Euler's Identity) (Euler's Pentagonal Number Theorem)

$$\prod_{i=1}^{\infty}(1-x^i) = 1 + \sum_{m=1}^{\infty} (-1)^m \left(x^{\frac{3m^2-m}{2}} + x^{\frac{3m^2+m}{2}} \right)$$

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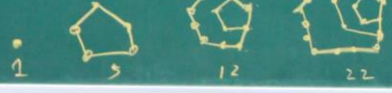

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Proof An elegant proof was due to Franklin (1881) but will not be given here.

Property Let $p(n) = 0$ for $n < 0$ and $p(0) = 1$.

$$p(n) = \sum_{m=1}^{\infty} (-1)^{m+1} \left[p\left(n - \frac{3m^2-m}{2}\right) + p\left(n - \frac{3m^2+m}{2}\right) \right].$$

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$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$$

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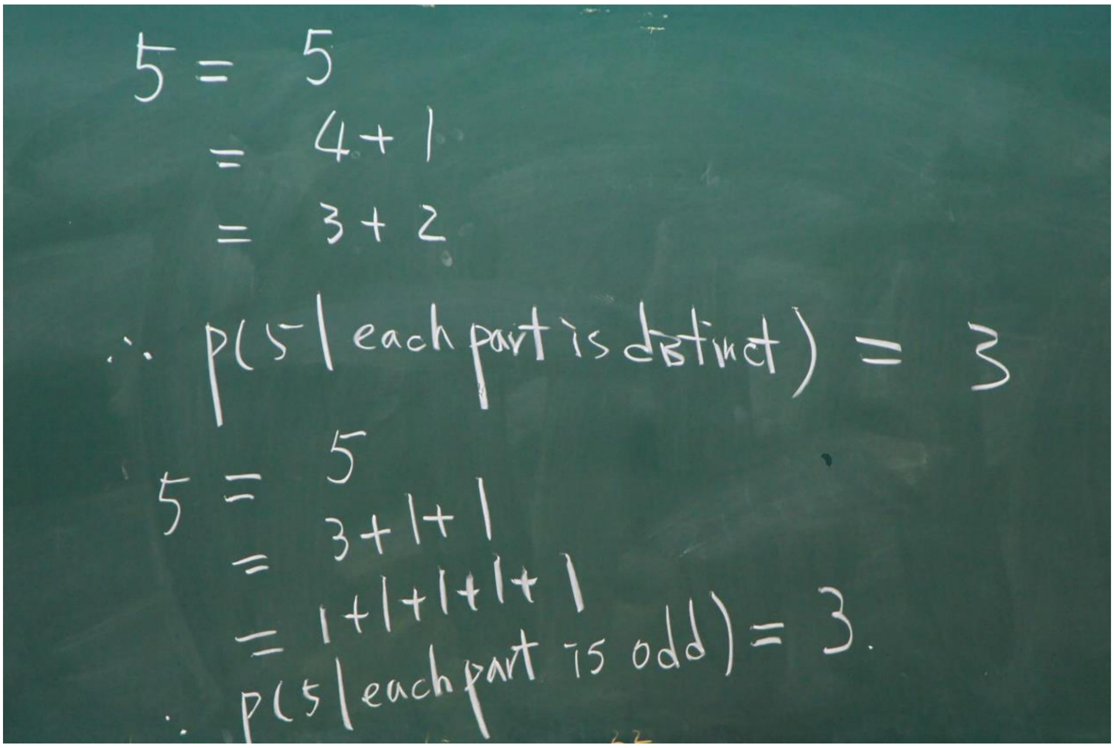
n	1	2	3	4	5	6	7	8	9	10	……	100	…
$p(n)$	1	2	3	5	7	11	15	22	30	42	……	190569292	…

Let $p(n \mid P)$ denote the number of partitions of n which satisfy the property P.

For example, $p(n \mid \text{each part is distinct})$,
$p(n \mid \text{each part is odd})$, ……


5 = 5
= 4 + 1
= 3 + 2
∴ p(5 | each part is distinct) = 3
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